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ON THE INVERSE PROBLEM OF NATURAL VIBRATIONS OF ELASTIC SHELLS

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The problem of determining small changes in the geometric parameters of elastic bodies is considered. It is assumed that the frequency spectrum of their natural vibrations should have given small changes. The method of the small parameter is applied, the problem is reduced to solving an l-moment problem. As an illustration, the problem of determining the variable stiffness of an elastic beam as well as the problem of determining the meridian shape of shells of revolution by means of given frequencies of natural vibrations are considered.

It should be mentioned that the most exhaustive results on such problems exist from the inverse Sturm-Liouville problem [1, 2] as well as for the inverse problem of quantum scattering theory [3, 4].

Only a few papers are devoted to inverse problems of elastic body vibrations. However, the problem of determining the density of an inhomogeneous string by means of its frequency spectra has been investigated with mathematical rigor [5–7]. The problem of determining the stiffness of a beam by means of given natural vibrations frequencies has been considered in an elementary formulation in [8]. This problem has been examined for beams and plates in more detail in [9, 10], where a method is given for the construction of the variable thickness for several given first natural vibrations frequencies and its numerical realization is demonstrated in examples. The present paper is a development of these others.

1. Formulation of the problem. Let us consider the following inverse natural vibrations problem resulting from the first part of [7], under the assumption of smallness in the increments of the natural frequencies.

Let there be the self-adjoint eigenvalue problem

$$Au - \lambda Bu = 0, \qquad G_i u = 0 \qquad (i = 1, \ldots, 2n) \tag{1.1}$$

where

$$Au = \sum_{i=0}^{n} (-1)^i [a_i(\alpha, x) u^{(i)}]^{(i)} \qquad Bu = \sum_{i=0}^{m} (-1)^i [b_i(\alpha, x) u^{(i)}]^{(i)}, \; m < n \tag{1.2}$$

and $G_j u$ denote the boundary conditions

$$u^{(k)} = 0 \quad (k = 0, 1, \ldots, n-1) \quad \text{for} \quad x = x_0, x_1 \tag{1.3}$$

For some function $\alpha(x) = \alpha_0(x)$ let the eigenvalues of the problem (1.1) be

$$0 < \lambda_{01} < \lambda_{02} < \ldots, \tag{1.4}$$

The problem is to determine the function $\alpha(x)$ so that the eigenvalues (1.4) would have given small changes, i.e. it is necessary to find the function $\alpha(x)$ by using the given eigenvalue spectrum of the problem (1.1)

$$\lambda_i = \lambda_{0i} + \varepsilon\lambda_{1i} \tag{1.5}$$

where ε is a small number.

Let us take ε as a small parameter, and let us assume that the function $\alpha(x)$, the operators A, B and the eigenfunctions u_i are expandable in series in this parameter

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + \ldots, \qquad u_i = u_{0i} + \varepsilon u_{1i} + \ldots$$
$$A = A_0 + \varepsilon A_1 + \ldots, \qquad B = B_0 + \varepsilon B_1 + \ldots \tag{1.6}$$

where

$$A_1 u = \sum_{i=0}^{n} (-1)^i \left[\alpha_1 \frac{\partial a_i(\alpha_0, x)}{\partial \alpha} u^{(i)}\right]^{(i)}, \quad B_1 u = \sum_{i=0}^{m} (-1)^i \left[\alpha_1 \frac{\partial b_i(\alpha_0, x)}{\partial \alpha} u^{(i)}\right]^{(i)} \tag{1.7}$$

Substituting (1.5), (1.6) into the equation and boundary conditions (1.1), we find that

$$A_0 u_{0i} - \lambda_{0i} B_0 u_{0i} = 0, \qquad G_j u_{0i} = 0 \tag{1.8}$$

$$A_0 u_{1i} - \lambda_{0i} B_0 u_{1i} = \lambda_{0i} B_1 u_{0i} + \lambda_{1i} B_0 u_{0i} - A_1 u_{0i}$$

$$G_j u_{1i} = 0 \qquad (i = 1, 2, \ldots; \ j = 1, \ldots, 2n) \tag{1.9}$$

Equations (1.9) connect the given natural frequencies λ_{1i} with the desired function $\alpha_1(x)$. These equations have nontrivial solutions when their right sides are orthogonal to the solutions of problem (1.8). Taking account of the normalization condition

$$\int_{x_0}^{x_1} u_{0i} B_0 u_{0i}\, dx = 1 \tag{1.10}$$

the orthogonality conditions become

$$\lambda_{1i} = \int_{x_0}^{x_1} (u_{0i} A_1 u_{0i} - \lambda_{0i} u_{0i} B_1 u_{0i})\, dx \tag{1.11}$$

Substituting A_1, B_1 into (1.11) according to the relationships (1.7), and taking account of the homogeneity of the boundary conditions (1.9), conditions (1.11) can be converted into

$$\lambda_{1i} = \int_{x_0}^{x_1} \alpha_1(x)\, g_i(x)\, dx \qquad (i = 1, 2, \ldots) \tag{1.12}$$

where

$$g_i(x) = \sum_{s=0}^{n} \frac{\partial a_s(\alpha_0, x)}{\partial \alpha} (u_{0i}^{(s)})^2 - \lambda_{0i} \sum_{s=0}^{m} \frac{\partial b_s(\alpha_0, x)}{\partial \alpha} (u_{0i}^{(s)})^2 \tag{1.13}$$

Moments of the desired functions $\alpha_1(x)$ relative to the functions $g_i(x)$ are given by the relationships (1.12). Let us assume the $g_i(x)$ to be linearly independent functions, to belong to the space of measurable and integrable-modulus functions with pth power L^p, and let us consider the function $\alpha_1(x)$ to belong to the space $L^{p'}$, where $1/p + 1/p' = 1$, and in addition to be bounded in the norm

$$\|\alpha_1(x)\| \leqslant l \tag{1.14}$$

Then the problem under consideration can be reduced to the solution of the l-moment problem [11, 12]. In this case the l-moment problem is to find the necessary and sufficient conditions for the existence of the function $\alpha_1(x) \in L^{p'}$ satisfying condition (1.14) and having the number λ_{1i} as their moments relative to the function $g_i(x)$.

Thus, the inverse problem of natural vibrations can be reduced, under definite assumptions, to the well worked out l-moment problem. In the following sections some examples are presented of the solution of specific problems of the vibrations of elastic bodies by such a method.

2. Inverse problem of free beam vibrations. Let us consider the problem of determining the size of similar cross sections of a hinge-supported beam according to its given frequencies of natural vibrations.

The differential equation for this problem can be represented as

$$(\alpha^2 y'')'' - \lambda \alpha y = 0 \tag{2.1}$$

where $\alpha(x)$ is the desired function characterizing the cross section. The boundary conditions are the following:

$$y(0) = y''(0) = y(1) = y''(1) = 0 \tag{2.2}$$

Let us proceed from the case of constant cross section $\alpha_0 = 1$. Then the eigenvalues and the normalized eigenfunctions of the problem (2.1), (2.2) are

$$\lambda_{0i} = i^4\pi^4, \qquad y_{0i} = \sqrt{2}\sin i\pi x \tag{2.3}$$

Let us now determine the function $\alpha(x)$ for the case when the first k eigenvalues differ slightly from the values (2.3), and the remaining eigenvalues agree with the values in (2.3), i.e. for the case

$$\lambda_i = i^4\pi^4 + \varepsilon_i \tag{2.4}$$

where ε_i are small numbers if $i \leqslant k$, and $\varepsilon_i = 0$, if $i > k$.

According to (1.13), (2.3) we have

$$g_i(x) = 2i^4\pi^4 \sin^2 i\pi x = i^4\pi^4(1 - \cos 2i\pi x) \tag{2.5}$$

Hence, in the case under consideration

$$\varepsilon_i = i^4\pi^4 \int_0^1 \alpha_1 (1 - \cos 2i\pi x)\, dx \qquad (i = 1, 2, \ldots) \tag{2.6}$$

Let us examine the problem in the space L^2 by appending the condition

$$\|\alpha_1\| = \left(\int_0^1 \alpha_1^2(x)\, dx\right)^{1/2} \leqslant l \tag{2.7}$$

to the relationships (2.6). It is known from the theory of the l-moment problem [12] that the necessary and sufficient condition for the solvability of the problem (2.6), (2.7) is the solvability of all the finite dimensional l-problems which are obtained from (2.6), (2.7) if $i = 1, \ldots, n$, where n is a fixed finite number. To solve this latter moments problem it is necessary and sufficient that there exist n numbers $\xi_1^\circ, \ldots, \xi_n^\circ$ yielding the solution of the following problem. Find

$$\min_{\xi_1,\ldots,\xi_n} \int_0^1 \left|\sum_{i=1}^n \xi_i g_i(x)\right|^2 dx = \int_0^1 \left|\sum_{i=1}^n \xi_i^\circ g_i(x)\right|^2 dx = \frac{1}{\Lambda^2} \geqslant \frac{1}{l^2} \tag{2.8}$$

under the condition

$$\xi_1\varepsilon_1+\ldots+\xi_n\varepsilon_n=1 \tag{2.9}$$

The solution $\alpha_1(x)$ of the finite-dimensional l-moment problem is hence expressed as

$$\alpha_1(x)=\Lambda^2\sum_{i=1}^{n}\xi_i^{\circ}g_i(x) \tag{2.10}$$

Taking account of (2.5), (2.6) and utilizing the Lagrange multiplier rule, the solution of the problem (2.8), (2.9) can be found as the solution of the system

$$\frac{\partial f(\xi)}{\partial\xi_i}=0, \qquad \sum_{i=1}^{k}\xi_i\varepsilon_i=1 \qquad (i=1,\ldots,n) \tag{2.11}$$

where

$$f(\xi)=\int_0^1\left[\sum_{i=1}^{n}\xi_i i^4\pi^4(1-\cos 2i\pi x)\right]^2dx-\mu\left(\sum_{i=1}^{k}\xi_i\varepsilon_i-1\right) \tag{2.12}$$

and μ is the Lagrange multiplier. It is easy to find that

$$\xi_i^{\circ}=\frac{\mu}{i^4\pi^4}\sum_{s=1}^{k}\left(\delta_{is}-\frac{2}{2n+1}\right)\frac{\varepsilon_s}{s^4\pi^4} \tag{2.13}$$

where

$$\mu=\left[\sum_{s=1}^{k}\left(\frac{\varepsilon_s}{s^4\pi^4}\right)^2-\frac{2}{2n+1}\left(\sum_{s=1}^{k}\frac{\varepsilon_s}{s^4\pi^4}\right)^2\right]^{-1} \tag{2.14}$$

If (2.13), (2.14) are utilized to evaluate the minimum value of the integral (2.8), there is obtained

$$\frac{1}{\Lambda^2}=\frac{1}{2}\mu\geqslant\frac{1}{l^2} \tag{2.15}$$

Substituting the found quantities (2.13), (2.15) into condition (2.10), we find the desired function

$$\alpha_1(x)=2\sum_{i=1}^{n}\sum_{s=1}^{k}\left(\delta_{is}-\frac{2}{2n+1}\right)\frac{\varepsilon_s}{s^4\pi^4}(1-\cos 2i\pi x) \tag{2.16}$$

Passing to the limit as $n\to\infty$, there results from (2.14)–(2.16) that if

$$l^2\geqslant 2\sum_{s=1}^{k}\left(\frac{\varepsilon_s}{s^4\pi^4}\right)^2 \tag{2.17}$$

then the inverse vibrations problem being examined has the solution

$$\alpha_1(x)=-2\sum_{s=1}^{k}\frac{\varepsilon_s}{s^4\pi^4}\cos 2s\pi x \tag{2.18}$$

This solution is unique in the case of equality in (2.17). The problem has a continuum of solutions in the inequality case.

3. Inverse problem of free membrane shell vibrations. As an illustration of the application of the proposed method to solve inverse free shell vibrations problems, let us consider the axisymmetric vibrations of an almost cyclindrical shell of revolution.

If the arclength is taken as coordinate on the middle surface, the problem is described by the following equations in dimensionless quantities:

$$T_1=u'-k_1w+\nu\frac{B'}{B}u-\nu k_2w, \qquad (BT_1)'-B'T_2+\lambda u=0 \tag{3.1}$$

$$T_2 = \frac{B'}{B} u - k_2 w + \nu u' - \nu k_1 w, \qquad k_1 T_1 + k_2 T_2 + \lambda w = 0 \qquad \text{(cont.)}$$

Here T_1, T_2 are stresses, u, w displacements, k_1, k_2 curvatures, B is the distance along the axis of rotation, and λ a frequency parameter.

Let us henceforth utilize the Gauss and Codazzi conditions for shells of revolution

$$Bk_1k_2 = -B'' \qquad (Bk_2)' = k_1 B' \tag{3.2}$$

Let us proceed from the solution of the problem for a cylindrical shell obtained from (3.1) for $B_0 = 1$, $k_{01} = 0$, $k_{02} = 1$

$$u_0'' - \nu w_0' + \lambda_0 u_0 = 0, \qquad -w_0 + \nu u_0' + \lambda_0 w_0 = 0 \tag{3.3}$$

Let the boundary conditions of the problem be

$$u_0(0) = 0, \qquad u_0(L) = 0 \tag{3.4}$$

The eigen modes and eigenvalues resulting from (3.3) and the boundary conditions (3.4) are

$$u_{0n} = C_n \sin s_n\beta, \qquad w_{0n} = \frac{\nu s_n}{1-\lambda_{0n}} C_n \cos s_n\beta \tag{3.5}$$

$$C_n^2 = 2L\left[L^2 + \left(\frac{\nu n\pi}{1-\lambda_{0n}}\right)^2\right]^{-1} \qquad s_n = \frac{n\pi}{L}$$

$$\lambda_{0n}^{\mp} = {}^1/_2(1+s_n^2) \pm {}^1/_2\sqrt{(1+s_n^2)^2 - 4(1-\nu^2)s_n^2} \tag{3.6}$$

Let us now find the shape of a shell of revolution such that its first eigenvalues λ_{0n}^{+} would differ from the values (3.6) by given small numbers ε_i. In the case under consideration $k_1(\beta)$, $K_2(\beta)$ and $B(\beta)$ would be the desired functions. But two of these functions can be expressed in terms of the third if the Gauss and Codazzi conditions (3.2) are used. Obtained from these conditions by the method of the small parameter is that the first members in an expansion in the small parameter are related by means of

$$k_{11} = -B_1'', \qquad k_{12} = -B_1 \tag{3.7}$$

Taking account of (3.7), the method described in Sect. 1 permits derivation of the following moments for the function B_1 in this case:

$$\varepsilon_i^* = \int_0^L B_1(\beta) g_i(\beta)\, d\beta \tag{3.8}$$

Here

$$\varepsilon_i^* = \varepsilon_i - 2C_i^2 \frac{\lambda_{0i}}{1-\lambda_{0i}} [B_1'(L) - B_1'(0)] \tag{3.9}$$

$(\varepsilon_i = 0$ for $i = k+1, k+2, \ldots)$

$$g_i(\beta) = 2[(u_{0i}' - \nu w_{0i}) w_{0i}]'' + \lambda_{0i}(u_{0i} w_{0i})' - 2\lambda_{0i}(u_{0i}^2 + w_{0i}^2) \tag{3.10}$$

In contrast to the preceding illustration, an additional term is contained in this problem which is due to the fact that the second derivatives $B_1''(\beta)$ of the desired function are introduced in the formulation of the problem by means of (3.7). The unknown factor $B_1'(L) - B_1'(0)$ in the additional term in the solution of the l-moment problem can be considered as a parameter whose value is determined after the problem has been solved, by using the found solution $B_1(\beta)$. By means of (3.5), (3.10) we have

$$g_i(\beta) = A_i - D_i \cos 2s_i\beta \tag{3.11}$$

$$A_i = \frac{(1-s_i^2)\lambda_{0i}}{2(1-\lambda_{0i})}, \qquad D_i = \frac{[(1+2\nu)s_i^2-1]\lambda_{0i}}{2(1-\lambda_{0i})} \tag{3.12}$$

Solving the n-dimensional l-moment problem (3. 8) under the condition

$$\int_0^L B_1^2(\beta)\,d\beta \leqslant l^2 \tag{3.13}$$

analogously to the preceding example, we have

$$\xi_i^0 = \frac{\mu}{LA_i}\sum_{s=1}^{n}\left\{\delta_{is} - 2\left[\gamma_{ii}\left(2\sum_{n=1}^{n}\frac{1}{\gamma_{rr}}+1\right)\right]^{-1}\right\}\frac{\varepsilon_s^*}{A_s\gamma_{ss}} \tag{3.14}$$

$$\mu = L\left[\sum_{i=1}^{n}\frac{\varepsilon_i^2}{D_i^2} - 2\left(2\sum_{n=1}^{n}\frac{1}{\gamma_{rr}}+1\right)^{-1}\left(\sum_{i=1}^{n}\frac{\varepsilon_i}{A_i\gamma_{ii}}\right)^2\right]^{-1} \tag{3.15}$$

It can be shown that (2. 15) remains valid in the case under consideration. Correspondingly

$$B_1(\beta) = \frac{2}{L}\sum_{i=1}^{n}\sum_{s=1}^{n}\left[\delta_{is} - \frac{2}{\gamma_{ii}}\left(2\sum_{n=1}^{n}\frac{1}{\gamma_{ii}}+1\right)^{-1}\right]\frac{\varepsilon_s^*}{A_s\gamma_{ss}}\left(1-\frac{D_i}{A_i}\cos 2s_i\beta\right) \tag{3.16}$$

It is seen from (3. 9), (3. 16) that

$$B_1'(L) = B_1''(0) = 0, \qquad \varepsilon_i^* = \varepsilon_i \tag{3.17}$$

Passing to the limit as $n\to\infty$ in the relationships (3. 15), (3. 16), we finally obtain that the considered inverse problem of shell vibrations has a solution if

$$\frac{2}{L}\sum_{i=1}^{k}\frac{\varepsilon_i^2}{D_i^2} \leqslant l^2 \tag{3.18}$$

The solution has the form

$$B_1(\beta) = -2\sum_{i=1}^{k}\frac{\varepsilon_i}{D_i}\cos 2s_i\beta \tag{3.19}$$

and is unique for the case of equality in the relationship (3. 18).

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ON THE DENSITY OF EIGENVALUES IN PROBLEMS OF STABILITY OF THIN ELASTIC SHELLS

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Asymptotic estimates are determined for the density of eigenvalues. The existence of points of concentration of the eigennumbers is established. Results for the natural frequencies of shells and the eigenvalues in stability problems are compared. Conditions are written down for the solvability of the linear equation describing the stability in the presence of small perturbations.

1. The stability equation for given stress resultants in the middle surface of a shell whose radii of curvature are almost constant is

$$D\Delta\Delta\Delta\Delta\Phi_{nm} + Eh\Delta_k\Delta_k\Phi_{nm} + \lambda_{nm}\Delta\Delta(\alpha_1\Phi_{,xx} + \alpha_2\Phi_{,yy}) = 0 \tag{1.1}$$

$$\alpha_1\lambda_{nm} = N_1, \quad \alpha_2\lambda_{nm} = N_2 \quad (0 \leqslant \alpha_1, \alpha_2 \leqslant 1), \qquad w = \Delta\Delta\Phi$$

$$\Delta\Phi = \Phi_{,xx} + \Phi_{,yy}, \quad \Delta_k\Phi = R_2^{-1}\Phi_{,xx} + R_1^{-1}\Phi_{,yy}, \quad \varphi = Eh\Delta_k\Phi$$

Here x, y are Cartesian coordinates, $w(x, y)$ is the normal shell deflection, $\varphi(x, y)$ is the stress function, Φ is the resolving function, D is the cylindrical stiffness, E, ν is the Young's modulus and Poisson's ratio, h = const is the shell thickness, $R_1 \approx$ const, $R_2 \approx$ const are the radii of curvature, $-N_1$ and $-N_2$ are two constant normal compressive forces.

A rectangular shell of nonnegative Gaussian curvature, hinge-supported along the sides is considered

$$0 \leqslant x \leqslant a, \qquad 0 \leqslant y \leqslant b$$

The shell buckling mode, to the accuracy of a normalized constant, is